On this lecture

Lecture Questions Welcome! Chat, Interrupt, End of lecture all OK

Exercises - Wed 2:15pm w/ Lucas Mann

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send

- Open for all
- Discusses material of lecture (Questions)

  & Weekly exercise sheet
- Invitation to hand in solutions for feedback

  (Voluntary, Bonn students only)
- Exam Admission info soon.

Course Content Ellipt. curves (1/3), Abelian Varieties (2/3)

Prereqs. Alg Geom (80%), Complex Analysis (20%)

§1 Definition k field.

Def Elliptic curve $/k \underset{def}{=}$ Grp. sch. $(E, m) / \operatorname{Spec} k$ w/
E proper smooth connected curve $/k$.

Explanation Grp Sch $/k \underset{def}{=}$ (G, m) where

·) G $\operatorname{Spec} k$ - scheme

·) $m : G \times_k G \longrightarrow G$ is a morphism
(the multiplication map)

s.th. $\forall\ S \to \operatorname{Spec} k$, the resulting map

$$G(S) \times G(S) = (G \times_k G)(S) \xrightarrow{m \circ -} G(S) \text{ makes}$$

$G(S)$ into a group.

Example 1) Γ group.

$\underline{\Gamma} := \coprod_{\gamma \in \Gamma} \operatorname{Spec} k$ w/ $\underline{\Gamma} \times_k \underline{\Gamma} = \coprod_{(\gamma_1, \gamma_2)} \operatorname{Spec} k \longrightarrow \underline{\Gamma}$

from $\Gamma \times \Gamma \longrightarrow \Gamma$.

Constant grp sch, non-connected if $\Gamma \neq \{1\}$,
0-dimensional

2) $(\mathbb{G}_m, m)$ w/ $\mathbb{G}_m = \operatorname{Spec} k[T^{\pm 1}]$

$$\begin{array}{rcl} m: \mathbb{G}_m \times_k \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ k[T^{\pm 1}] \otimes_k k[T^{\pm 1}] & \longleftarrow & k[T^{\pm 1}] \\ T \otimes T & \longleftarrow\!\shortmid & T \end{array}$$

Then $(\mathbb{G}_m, m)(\operatorname{Spec} R) = R^\times$.

$\mathbb{G}_m$ <u>multiplicative grp</u>, connected, smooth, one-dim, affine, not proper $/k$.

Nice feature: $(\mathbb{G}_m, m)$ is <u>commutative</u> grp sch, i.e.

$$\begin{tikzcd} \mathbb{G}_m \times_k \mathbb{G}_m \arrow[rr, "\text{switch}"] \arrow[dr, "m"'] & \circlearrowleft & \mathbb{G}_m \times_k \mathbb{G}_m \arrow[dl, "m"] \\ & \mathbb{G}_m & \end{tikzcd}$$

Equivalently: $(\mathbb{G}_m, m)(S) = \mathcal{O}_S(S)^\times$ commutative $\forall S/k$.

3) Exercise: Define $m: GL_n \times GL_n \longrightarrow GL_n$ where

$$GL_n = \operatorname{Spec} k[T_{ij}, S]_{i,j=1}^{n} \big/ (S \cdot \det(T_{ij}) - 1)$$

s.th. $(GL_n, m)(\operatorname{Spec} R) = GL_n(R)$ as group.

Then $GL_n \longrightarrow \operatorname{Spec} k$ connected, smooth, n^2-dim, <u>not commutative</u>.

You noticed: Useful to think about $R^\times$, $GL_n(R)$, ...

in addition to $\mathbb{G}_m$, GL_n, ...

Two perspectives equivalent by

<u>Prop (Yoneda-Lemma)</u> $\exists$ natural fully faithful

$$Sch/_k \hookrightarrow Fun(Sch/_k^{op}, Set)$$

$$T \longmapsto [h_T : S \mapsto T(S) := Mor_k(S, T)]$$

<u>E.g.</u>
$$\mathbb{G}_m \longmapsto [h_{\mathbb{G}_m} : S \mapsto \mathcal{O}_S(S)^\times]$$

$$GL_n \longmapsto [h_{GL_n} : S \mapsto GL_n(S)]$$

For (G, m) grp sch, Yoneda implies existence of (uniquely determined) additional structure:

<u>Neutral Element</u>: $h_{\operatorname{Spec} k}(S) = \{*\} \ni * \longmapsto e_S \in h_G(S)$

comes from $e : \operatorname{Spec} k \to G$ uniquely det neutral element

<u>Inversion</u> $h_G(S) \ni g \longmapsto g^{-1} \in h_G(S)$

comes from $\iota : G \to G$

<u>n-th power</u> $g \mapsto g^n$ from $[n] : G \to G$.

§2 ECs are commutative

<u>Prop</u> An ellipt. curve is a commutative grp. sch.

<u>Proof</u> Given (E, m), consider conj. morphism

$c : E \times E \longrightarrow E, \quad (h,g) \longmapsto hgh^{-1}.$

Then E commutative $\Leftrightarrow$ $c = pr_2$

Let $e \in E(k)$ be neutral element.

Since source & target are integral, enough to show after any localization, e.g. for

$$E \times \operatorname{Spec} \mathcal{O}_{E,e} \longrightarrow \operatorname{Spec} \mathcal{O}_{E,e}$$

Since $\mathcal{O}_{E,e} \hookrightarrow (\mathcal{O}_{E,e})^{\wedge}_{m_e}$ ⟵ m_e-adic completion,

enough to consider all

$$E \times X_n \xrightarrow{c_n} X_n, \quad n \geq 1, \quad X_n := \operatorname{Spec} V_n$$

$$V_n := \mathcal{O}_{E,e} / m_e^n.$$

Now $E \times X_n \longrightarrow \operatorname{Spec} k$ is a proper scheme w/

$$H^0(E \times X_n, \mathcal{O}_{E \times X_n}) = V_n.$$

Then c_n given by map of k-algebras $V_n \xleftarrow{c_n^*} V_n$.

Since composition $\operatorname{Spec} k \times_k X_n \xrightarrow{(e, id)} E \times X_n \longrightarrow X_n$

is identity, c_n^* is identity and hence $c_n = pr_2$ $\square$

§3 Riemann Surfaces.

<u>Recall</u> Function $\mathbb{C} \supseteq U \xrightarrow{f} \mathbb{C}$ <u>holomorphic</u> if

it is smooth as fct $\mathbb{R}^2 \supseteq U \xrightarrow{(f_1, f_2)} \mathbb{R}^2$ and if its

Jacobian $\begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}$ is $\mathbb{C}$-linear. $\left(x + iy \mapsto (x, y) \right)$.

<u>Equivalent</u> $\forall a \in U$, $f(z) = \sum_{n \geq 0} c_n \cdot (z-a)^n$

has a convergent p.s.-expansion on open nbhd of a.

Definition is purely local, so holom. fcts. form sheaf $\mathcal{O}_U$.

By p.s.-characterization, $(U, \mathcal{O}_U)$ is <u>locally</u> ringed space

(max ideal in $\mathcal{O}_{U,a}$ is $(z-a) \cdot \mathcal{O}_{U,a}$)

<u>Def</u> Riemann surface $\underset{\text{def}}{=}$ Loc ringed sp $(X, \mathcal{O}_X)$

that is locally $\cong (U, \mathcal{O}_U)$ for $U \subseteq \mathbb{C}$ open.

(Often Hausdorff, 2^{nd} countable, ... are added.)

Construction $C/\operatorname{Spec}\mathbb{C}$ smooth separated curve.

$\implies X := C(\mathbb{C})$ is naturally a R.S.

Sketch 1) If $C = \operatorname{Spec} A$, choose $\mathbb{C}[T_1, \dots, T_r] \twoheadrightarrow A$,

endow $X \underset{T_i}{\subseteq} \mathbb{C}^r$ w/ subspace topology.

Claim Is indep of coordinates.

Pf

$$\begin{tikzcd} & \mathbb{C}[T_1, \dots, T_r] \arrow[dl, two heads] \arrow[dd, bend left, "\psi"] \\ A & \\ & \mathbb{C}[S_1, \dots, S_s] \arrow[ul, two heads] \arrow[uu, bend left, "\exists \varphi"] \end{tikzcd}$$

Get

$$\begin{tikzcd} X \underset{}{\overset{T_i}{\subseteq}} \mathbb{C}^r \arrow[d, bend left, "\varphi, \psi"] \\ X \underset{S_j}{\subseteq} \mathbb{C}^s \arrow[u, bend left] \end{tikzcd}$$

Then φ, ψ are continuous (since given by polynomials) and are identity on X. $\square$

2) For general C, separatedness $\Longrightarrow$ intersect of affines is affine.

$\Longrightarrow$ may glue 1) to make X into top sp.

3) $\mathcal{O}_X \subseteq \mathrm{Cont}(X, \mathbb{C})$ subsheaf of

$f : U \to \mathbb{C}$ s.th. $\forall\, x \in U$, there are open nbhd $x \in U_x \subseteq U$ + $g \in \mathcal{O}_{C,x}$ defined on U_x

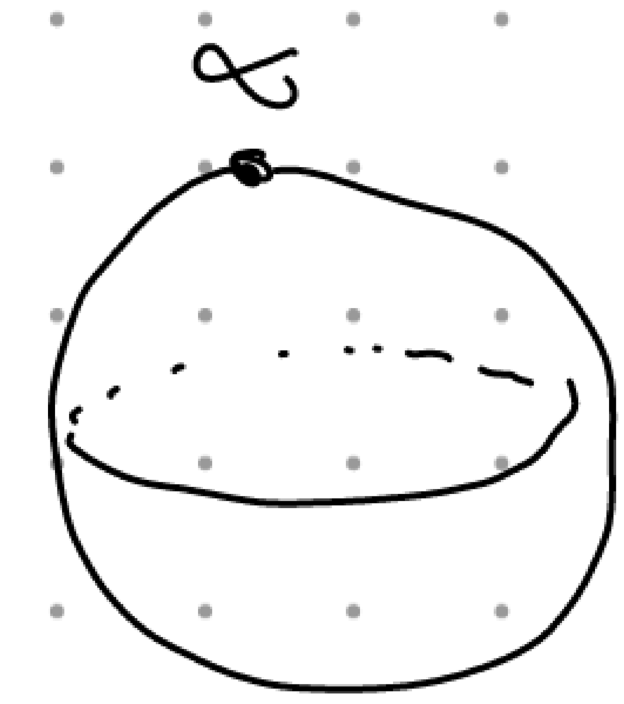
s.th. $f|_{U_x} = \varphi \circ g$ for holom fct φ.

(In other words, the local uniformizers $g_x \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$ provide local coordinates on X. This works b/c of the power series charact. of holom. fcts.)

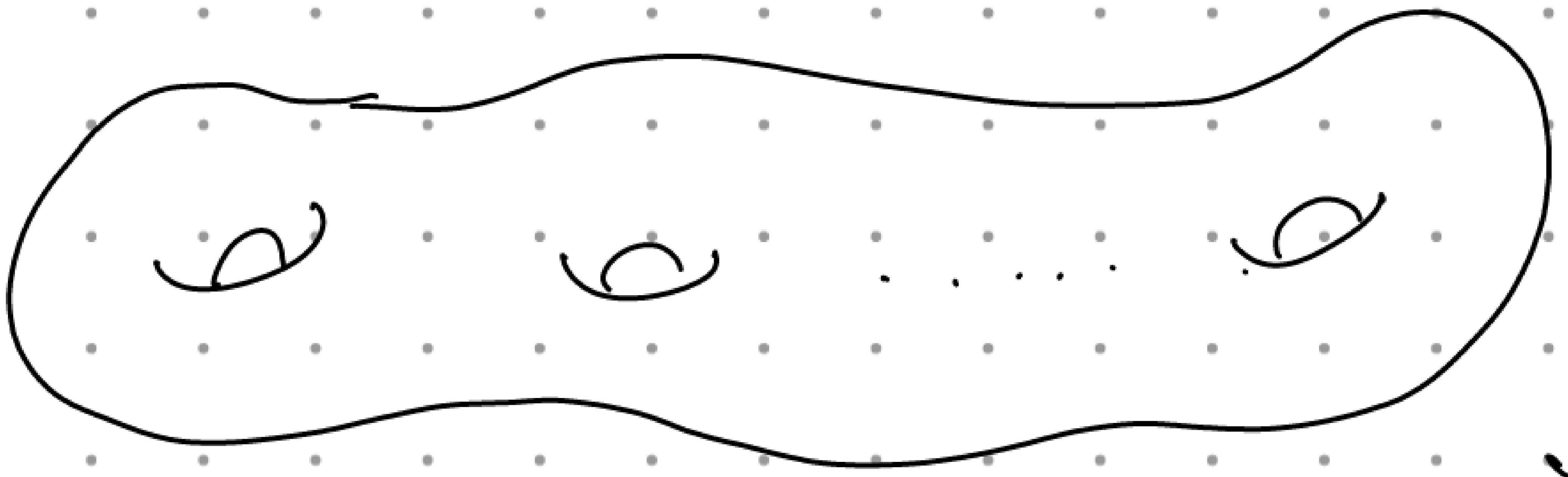
<u>Example</u> $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ glued along $z \mapsto z^{-1}$

$\Longrightarrow$ $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \mathbb{C}$ $\quad$ ——— $\cup$ ———

$=$ Riemann sphere



Intuition for compact case: Underlying top space is



rings = genus (X)

(3g-3)-dimensional space of complex structures ($g \geq 2$)

<u>Thm</u> (Recap / Black Box) Have equiv of cats

$$\{\text{smooth proper } C/\operatorname{Spec}\mathbb{C}\} \xrightarrow{\sim} \{\text{Compact Riemann Surfaces } X\}$$

$$C \longmapsto C(\mathbb{C})$$

<u>Rmk</u> Fully faithfulness is doable, while Essential surjectivity is a real theorem!

Difficulty is to find <u>a single</u> non-constant merom fct on X.

§4 Elliptic curves / $\mathbb{C}$

<u>Thm</u> Every elliptic curve $E/\operatorname{Spec}\mathbb{C}$ is of the form $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for a lattice $\Lambda \subseteq \mathbb{C}$, group structure being std additive structure.

(Lattice = discrete additive subgroup $\Lambda \subseteq \mathbb{C}$ of $\mathbb{Z}$-rk 2)

In particular, E is of genus 1. 

<u>Rmk</u> Conversely any $\mathbb{C}/\Lambda$ is ellipt curve through previous equiv of cats.

<u>Proof</u> $X := E(\mathbb{C})$.

$L \xrightarrow{\pi} X$ universal cover.

Choose $\begin{matrix} L & & X \\ \Psi & & \Psi \\ 0 & \longmapsto & 0 \end{matrix}$, lift of neutral element.

<u>Claim</u> There is a unique abelian grp str on L w/ zero 0 s.th. π is a group homo.

<u>Pf</u> Given $x, y \in L$, choose $p, q : [0,1] \longrightarrow L$ s.th.

$$p(0) = q(0) = 0, \quad p(1) = x, \; q(1) = y.$$

Then $\overline{\tau} : [0,1] \to E$, $t \mapsto \pi(p(t)) + \pi(q(t))$

has a unique lift $r : [0,1] \to L$ s.th. $r(0) = 0$.

<u>Define</u> $x + y := r(1)$.

Since $x+y \in \pi^{-1}(\pi(x) + \pi(y))$, which is discrete,

only depends on p, q up to homotopy.

<u>Omitted</u> Group relations lift from E. (Try yourself!)

Then, since $\pi(x+y) = \pi(x) + \pi(y)$, claim is $\square$.

•) $\ker \pi$ acts by translation as Deck transformations,

$$\ker \pi \longrightarrow \mathrm{Aut}(L/X) \cong \pi_1(X, 0)$$

•) Action transitive on $\pi^{-1}(0)$, so $\ker \pi \cong \pi_1(X, 0)$.

•) Since $\pi_1(\mathbb{P}^1(\mathbb{C})) = 0$, $\pi_1($genus ≥ 2 surface$)$ non-abelian,

implies $L \cong \mathbb{C}$ as R.S., $\Lambda := \ker \pi \cong \mathbb{Z}^2$

and $\Lambda \subseteq \mathbb{C}$ discrete for <u>some</u> abelian grp str on $\mathbb{C}$.

<u>Claim</u> Grp str is std additive one.

Prf $\forall a \in \mathbb{C}$, translation

is a holomorphic bijective map.

From complex analysis: $a \bullet z = a + m_a \cdot z$ is linear $\forall a$.

Then $a \bullet z = a + m_a z = z + m_z a$

so $m_z = a^{-1}(a + m_a z - z)$ $\forall z$ if $a \neq 0$.

Since necessarily $m_z \neq 0$, get $m_a = 1$ $\forall a$. $\square$.